First semestral exam 2010
M.Math.IInd year
Advanced Number Theory — B.Sury
Answer 6 questions including the last one.

Q 1.
Give an example to show the tightness of Chevalley-Warning theorem; that is, show that a homogeneous polynomial of degree d in d variables over a finite field may not have nontrivial zeroes.
Hint: Look at the norm map from $\mathbf{F}_{q^d}$ to $\mathbf{F}_q$.

Q 2.
Prove that $2^n - 1$ does not divide $3^n - 1$ for $n > 1$.
Hint: Use quadratic reciprocity.

OR

Let $p \equiv 3 \bmod 8$ be a prime such that $(p-1)/2$ is a prime. Prove that $\mathbf{Z}_p^* =< 2 >$.
Hint: Look at the order of 2.

Q 3.
Given a Fermat prime $p = 2^{2^n} + 1$, determine (with proof) all the r such that $\mathbf{Q}_p$ contains a nontrivial r-th root of unity.

OR

Let $f = a_0 + a_1 X + \cdots + a_n X^n \in \mathbf{Q}_p[X]$ be irreducible of degree $n \geq 1$, where $f(0) \neq 0$. Use Hensel's lemma to prove that if $a_0, a_n \in \mathbf{Z}_p$, then $f \in \mathbf{Z}_p[X]$.

Q 4.
(i) Prove that the image of the 'n-th power map' on $\mathbf{Q}_p^*$ is open.
(ii) Give an example of a nontrivial homomorphism χ from $\mathbf{Q}_p \to \mathbf{C}^*$.
Hint for (ii): Think of the 'negative part' of any p-adic number.

Q 5.
(i) If a quadratic form over $\mathbf{Q}$ is isotropic, prove that it represents all values in $\mathbf{Q}$.
(ii) For any odd prime p, find all the anisotropic quadratic forms of rank 4 over $\mathbf{Q}_p$ up to isometry.

Q 6.
Let p be an odd prime and a, b, c integers such that $(p, abc) = 1$. Use Hensel's lemma (in several variables) to prove that $aX^2 + bY^2 + cZ^2 = 0$

has a non-trivial solution in $\mathbf{Q}_p$.

Q 7.

If an integer a is a square modulo p for all primes bigger than 10^{10}, prove that a must be a perfect square.
Hint: Apply the density version of Dirichlet's theorem on primes in progressions.

OR

(i) If $f(s) = \sum_{n\geq 1} a_n n^{-s} = \zeta(s)\zeta(s-2)$, determine the abscissa of convergence for Re f. Find an expression for $a(n)$.
(ii) Define the Liouville function as $\lambda(\prod_i p_i^{u_i}) = (-1)^{\sum u_i}$. Find the value of $\sum_{d|n} \lambda(d)$ and deduce that $\sum_n \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$.

Q 8.

Let L be a lattice in $\mathbf{C}$ with a basis v_1, v_2. Let p be a prime. Show that any sublattice of L of index p must be one of the lattices $L_i = \mathbf{Z}(v_1 + iv_2) + \mathbf{Z}pv_2 (0 \leq i < p)$ and $L_p := \mathbf{Z}pv_1 + \mathbf{Z}v_2$.
Hint: Establish a bijection between the set of matrices
$\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \geq 1, 0 \leq b < d, ad = p\}$ and the set of sublattices of index p.

OR

Argue to prove that $E_{12} - E_6^2 = c\Delta$ for some constant c. Determine c and deduce an expression for $\tau(n)$ in terms of $\sigma_{11}(n)$ and $\sigma_5(n)$.